Simpson type integral inequalities for harmonic convex functions via Riemann-Liouville fractional integrals

Zeynep Şanlı

Department of Mathematics, Faculty of Sciences, Karadeniz Technical University

E-mail: z.akdemirci@gmail.com

Abstract

The main aim of this investigation is to establish a new Simpson type Riemann-Liouville fractional integral equality for harmonically convex functions. Using this identity, some new results related to Simpson-like type Riemann-Liouville fractional integral inequalities are obtained. Then, some interesting conclusions are attained for some special cases of Riemann-Liouville fractional integrals when $\alpha = 1$.

2000 Mathematics Subject Classification. **26D15, 26D10**. 34A08.
Keywords. Simpson inequality, harmonic convex, integral equality.

1 Introduction

Convex functions play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. Because of this importance, convex functions have attracted the attention of many researchers. There are several modern works on convexity that are for the investigations of real analysis, functional analysis, linear algebra, and geometry. Numerous articles have been composed by various mathematicians on convex functions and inequalities for their different classes, see the references [15, 16].

In the other hand, in mathematics, inequalities are used to compare the relative size of values. They can be used to compare integers, variables, and various other algebraic expressions. This topic is widely studied in the literature. Some of the most studied inequalities are Hermite-Hadamard inequality, Ostrowski-type inequalities, Simpson-type inequalities etc.

The following inequality is known as Simpson's integral inequality:

Theorem 1.1. Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup\left|f^{(4)}(x)\right| < \infty$. Then, the following inequality holds:

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty . (b-a)^4 . \tag{1.1}$$

The study of Simpson type integral inequalities involving various kinds of convex functions has been carried out by many researchers, including Alomari et al. [7] in the study of Simpson's inequalities for s-convex functions, Matloka [9] in generalization of Simpson type inequality for extended h-convex functions, Rashid et al. [14] in the study of Simpson's type integral inequalities for k-fractional integrals and their applications, Sarikaya [13] in the research of generalized Simpson

Tbilisi Mathematical Journal Special Issue (8 - 2021), pp. 167–175.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 12 June 2020.
Accepted for publication: 22 November 2020.

type integral inequalities. For more results and recent development on the Simpson's inequality, see [4, 8, 10, 11, 12, 17].

In [6], İşcan gives the definition of harmonically convex functions as follows:

Definition 1.2. [6].Let $A \subset \mathbb{R}\backslash\{0\}$ and $f : A \to \mathbb{R}$ be a function. f is harmonically convex, if

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b) + (1-t)f(a) \tag{1.2}$$

for all $a, b \in A$ and $t \in [0,1]$. Otherwise, f is harmonically concave.

Harmonic convex functions are important for mathematical inequalities. Many authors obtain several inequalities for harmonic convex functions [6, 2, 18]. One of the most studied inequalities for harmonic convex functions is Hermite-Hadamard, which is stated as follows:

Theorem 1.3. [6]Let $f : A \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in A$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.3}$$

The motivation of this investigation is to establish Simpson type Riemann-Liouville fractional integral inequalities. Unlike studies in the literature, harmonic convex functions are used in this study and interesting results are obtained.

2 Preliminaries

In this section, we mention some definitions and fundamental results before our main results.

Definition 2.1. Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a,b]$. The left and right Riemann- Liouville fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ (see [1], p. 69).

3 Main results

Lemma 3.1. Let $f : I \subset (0, \infty) \to \mathbb{R}$, be a differentiable function on $I^\circ, a, b \in I^\circ$ and $a < b$. If $f' \in L[a, b]$, then the following equality holds:

$$I(a, b; \alpha) = \frac{1}{6}\left[f(a) + 4f\left(\frac{2ab}{a+b}\right) + f(b)\right] \tag{3.1}$$
$$- \frac{1}{2}\left(\frac{2ab}{b-a}\right)^\alpha \Gamma(\alpha+1)\left(J^\alpha_{\frac{1}{a}-}(f \circ h)\left(\tfrac{a+b}{2ab}\right) + J^\alpha_{\frac{1}{b}+}(f \circ h)\left(\tfrac{a+b}{2ab}\right) \right)$$
$$= \frac{b-a}{2ab}\int_0^1 \left(\tfrac{1}{3} - \tfrac{t^\alpha}{2}\right)\left[\begin{array}{l} \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^2 f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right) \\ -\left(\frac{2ab}{(1+t)a+(1-t)b}\right)^2 f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right) \end{array}\right] dt$$

and $h(x) = \frac{1}{x}, \alpha > 0$.

Proof. We start by considering the following computations which follows from change of variables and using the definition of the Riemann-Liouville fractional integrals.

$$I_1 = \int_0^1 \left(\frac{1}{3} - \frac{t^\alpha}{2}\right)\left(\frac{2ab}{(1-t)a + (1+t)b}\right)^2 f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right) dt$$
$$= \frac{2ab}{a-b}\left(-\frac{1}{6}f(a) - \frac{1}{3}f\left(\frac{2ab}{a+b}\right)\right) + \frac{2ab\alpha}{2(a-b)}\int_0^1 t^{\alpha-1} f\left(\frac{2ab}{(1-t)a+(1+t)b}\right) dt$$
$$= \frac{2ab}{a-b}\left(-\frac{1}{6}f(a) - \frac{1}{3}f\left(\frac{2ab}{a+b}\right)\right) + \frac{2ab\alpha}{2(a-b)}\left(\frac{2ab}{a-b}\right)^\alpha \Gamma(\alpha) J^\alpha_{\frac{1}{a}-}(f \circ h)\left(\tfrac{a+b}{2ab}\right)$$
$$= \frac{2ab}{a-b}\left(-\frac{1}{6}f(a) - \frac{1}{3}f\left(\frac{2ab}{a+b}\right)\right) + \frac{1}{2}\left(\frac{2ab}{a-b}\right)^{\alpha+1} \Gamma(\alpha+1) J^\alpha_{\frac{1}{a}-}(f \circ h)\left(\tfrac{a+b}{2ab}\right)$$

and similarly

$$I_2 = \int_0^1 \left(\frac{t^\alpha}{2} - \frac{1}{3}\right)\left(\frac{2ab}{(1+t)a + (1-t)b}\right)^2 f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right) dt$$
$$= \frac{2ab}{b-a}\left(\frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{2ab}{a+b}\right)\right) - \frac{1}{2}\left(\frac{2ab}{b-a}\right)^{\alpha+1} \Gamma(\alpha+1) J^\alpha_{\frac{1}{b}+}(f \circ h)\left(\tfrac{a+b}{2ab}\right).$$

Thus, we have

$$\frac{b-a}{2ab}(I_2 - I_1) = \frac{1}{6}\left[f(a) + 4f\left(\frac{2ab}{a+b}\right) + f(b)\right]$$
$$- \frac{1}{2}\left(\frac{2ab}{b-a}\right)^\alpha \Gamma(\alpha+1)\left(J^\alpha_{\frac{1}{a}-}(f \circ h)\left(\tfrac{a+b}{2ab}\right) + J^\alpha_{\frac{1}{b}+}(f \circ h)\left(\tfrac{a+b}{2ab}\right) \right).$$

Q.E.D.

Remark 3.2. If we take $\alpha = 1$ in Lemma 3.1, we have the following equality

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2ab}\right)+f(b)\right]-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \tag{3.2}$$
$$=\frac{b-a}{2ab}\int_0^1\left(\frac{1}{3}-\frac{t}{2}\right)\left[\begin{array}{c}\left(\frac{2ab}{(1-t)a+(1+t)b}\right)^2 f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\\ -\left(\frac{2ab}{(1+t)a+(1-t)b}\right)^2 f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\end{array}\right]dt.$$

Theorem 3.3. Let $f : I \subset (0,\infty) \to \mathbb{R}$, be a differentiable function on $I^{\circ}, a, b \in I^{\circ}$ and $a < b$. If $f' \in L[a,b]$ and $|f'|$ is a harmonic convex function on $[a,b]$, then the following inequality holds:

$$|I(a,b;\alpha)| \le \frac{b-a}{6ab}\left(\begin{array}{c}\left|f'(a)\right|\left(K_1(t;\alpha)+K_2(t;\alpha)\right)\\ +\left|f'(b)\right|\left(K_3(t;\alpha)+K_4(t;\alpha)\right)\end{array}\right) \tag{3.3}$$

where

$$K_1(t;\alpha)=\frac{2a^2b^2\left(2b\ln(|2b|)+2a\right)}{2b(b-a)^2}-\frac{2a^2b^2\left((b+a)\ln(|b+a|)\right)+2a}{(a+b)(b-a)^2},$$

$$K_2(t;\alpha)=-\frac{2a^2b^2\left(2a\ln(|2a|)+2a\right)}{2a(b-a)^2}+\frac{2a^2b^2\left((b+a)\ln(|b+a|)\right)+2a}{(a+b)(b-a)^2},$$

$$K_3(t;\alpha)=-\frac{2a^2b^2\left(2b\ln(|2b|)+2b\right)}{2b(b-a)^2}+\frac{2a^2b^2\left((b+a)\ln(|b+a|)\right)+2b}{(a+b)(b-a)^2},$$

$$K_4(t;\alpha)=\frac{2a^2b^2\left(2a\ln(|2a|)+2b\right)}{2a(b-a)^2}-\frac{2a^2b^2\left((b+a)\ln(|b+a|)\right)+2b}{(a+b)(b-a)^2},$$

$\alpha > 0$.

Proof. From Lemma 3.1 and $|f'|$ is harmonic convex, we have

$$|I(a,b;\alpha)| \le \frac{b-a}{2ab}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha}}{2}\right|\left(\begin{array}{c}\left(\frac{2ab}{(1-t)a+(1+t)b}\right)^2 f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\\ +\left(\frac{2ab}{(1+t)a+(1-t)b}\right)^2 f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\end{array}\right)dt$$
$$\le \frac{b-a}{2ab}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha}}{2}\right|\left(\begin{array}{c}\left(\frac{2ab}{(1-t)a+(1+t)b}\right)^2\left(\frac{1+t}{2}\left|f'(a)\right|+\frac{1-t}{2}\left|f'(b)\right|\right)\\ +\left(\frac{2ab}{(1+t)a+(1-t)b}\right)^2\left(\frac{1-t}{2}\left|f'(a)\right|+\frac{1+t}{2}\left|f'(b)\right|\right)\end{array}\right)dt$$
$$\le \frac{b-a}{6ab}\left(\begin{array}{c}\left|f'(a)\right|\left(K_1(t;\alpha)+K_2(t;\alpha)\right)\\ +\left|f'(b)\right|\left(K_3(t;\alpha)+K_4(t;\alpha)\right)\end{array}\right)$$

where we used the fact that $\left|\frac{1}{3}-\frac{t^{\alpha}}{2}\right| \le \frac{1}{3}$ for all $t \in [0,1]$.This completes the proof. Q.E.D.

Theorem 3.4. Let $f : I \subset (0, \infty) \to \mathbb{R}$, be a differentiable function on $I^\circ, a, b \in I^\circ$ and $a < b$. If $f' \in L[a, b]$ and $|f'|^q$ is a harmonic convex function on $[a, b]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$|I(a,b;\alpha)| \leq \frac{b-a}{6ab}\left[\begin{array}{c} \left(X_1(q;a,b)\left|f'(a)\right|^q + X_2(q;a,b)\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(X_3(q;a,b)\left|f'(a)\right|^q + X_4(q;a,b)\left|f'(b)\right|^q\right)^{\frac{1}{q}} \end{array}\right]. \tag{3.4}$$

where

$$X_1(q;a,b) = \int_0^1 \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^{2q} \frac{1+t}{2} dt,$$

$$X_2(q;a,b) = \int_0^1 \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^{2q} \frac{1-t}{2} dt,$$

$$X_3(q;a,b) = \int_0^1 \left(\frac{2ab}{(1+t)a+(1-t)b}\right)^{2q} \frac{1-t}{2} dt,$$

$$X_4(q;a,b) = \int_0^1 \left(\frac{2ab}{(1+t)a+(1-t)b}\right)^{2q} \frac{1+t}{2} dt,$$

$\alpha > 0$.

Proof. From Lemma 3.1 and using the Hölder's integral inequality and the harmonic convexity of $|f'|^q$, we have

$$|I(a,b;\alpha)| \leq \frac{b-a}{2ab}\left(\int_0^1 \left|\frac{1}{3} - \frac{t^\alpha}{2}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times \left\{\begin{array}{c} \left(\int_0^1 \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^{2q} \left|f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 \left(\frac{2ab}{(1+t)a+(1-t)b}\right)^{2q} \left|f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right\}$$

$$\leq \frac{b-a}{6ab}\left[\begin{array}{c} \left(\int_0^1 \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^{2q} \left[\begin{array}{c} \left|f'(a)\right|^q \left(\frac{1+t}{2}\right) \\ +\left|f'(b)\right|^q \left(\frac{1-t}{2}\right) \end{array}\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 \left(\frac{2ab}{(1+t)a+(1-t)b}\right)^{2q} \left[\begin{array}{c} \left|f'(a)\right|^q \left(\frac{1-t}{2}\right) \\ +\left|f'(b)\right|^q \left(\frac{1+t}{2}\right) \end{array}\right] dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq \frac{b-a}{6ab}\left[\begin{array}{c} \left(X_1(q;a,b)\left|f'(a)\right|^q + X_2(q;a,b)\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(X_3(q;a,b)\left|f'(a)\right|^q + X_4(q;a,b)\left|f'(b)\right|^q\right)^{\frac{1}{q}} \end{array}\right].$$

This completes the proof. Q.E.D.

Remark 3.5. If we take $\alpha = 1$ in Theorem 3.4, we obtain the following inequality

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{2ab}{a+b}\right) + f(b)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx \right|$$
$$\leq \frac{b-a}{6ab}\left[\begin{array}{c} \left(X_1(q;a,b)\left| f'(a)\right|^q + X_2(q;a,b)\left| f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(X_3(q;a,b)\left| f'(a)\right|^q + X_4(q;a,b)\left| f'(b)\right|^q\right)^{\frac{1}{q}} \end{array}\right].$$

Theorem 3.6. Let $f : I \subset (0,\infty) \to \mathbb{R}$, be a differentiable function on $I^\circ, a, b \in I^\circ$ and $a < b$. If $f' \in L[a,b]$ and $|f'|^q$ is a harmonic convex function on $[a,b]$ for $q \geq 1$, then the following inequality holds:

$$|I(a,b;\alpha)| \leq \frac{b-a}{2ab}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\begin{array}{c} \left(X_1(q;a,b)\left| f'(a)\right|^q + X_2(q;a,b)\left| f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(X_3(q;a,b)\left| f'(a)\right|^q + X_4(q;a,b)\left| f'(b)\right|^q\right)^{\frac{1}{q}} \end{array}\right] \tag{3.5}$$

where $X_1(q;a,b)$, $X_2(q;a,b)$, $X_3(q;a,b)$, $X_4(q;a,b)$ are defined as in Theorem 3.4 and $\alpha > 0$.

Proof. From Lemma 3.1 and using the power mean inequality, we have that the following inequality holds:

$$|I(a,b;\alpha)| \leq \frac{b-a}{2ab}\int_0^1 \left|\frac{1}{3} - \frac{t^\alpha}{2}\right| \left(\begin{array}{c} \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^2 \left| f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right| \\ +\left(\frac{2ab}{(1+t)a+(1-t)b}\right)^2 \left| f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right| \end{array}\right) dt$$

$$\leq \frac{b-a}{2ab}\left\{ \begin{array}{c} \left(\int_0^1 \left|\frac{1}{3} - \frac{t^\alpha}{2}\right| dt\right)^{1-\frac{1}{q}} \\ \times \left[\begin{array}{c} \left(\int_0^1 \left|\frac{1}{3} - \frac{t^\alpha}{2}\right| \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^{2q} \left| f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 \left|\frac{1}{3} - \frac{t^\alpha}{2}\right| \left(\frac{2ab}{(1+t)a+(1-t)b}\right)^{2q} \left| f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right] \end{array}\right\}.$$

By the harmonic convexity of $|f'|^q$ and the fact that $\left|\frac{1}{3} - \frac{t^\alpha}{2}\right| \leq \frac{1}{3}$ for all $t \in [0,1]$, we get

$$\begin{aligned}&\int_0^1 \left|\frac{1}{3}-\frac{t^\alpha}{2}\right| \left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)^{2q} \left| \ f'\left(\tfrac{2ab}{(1-t)a+(1+t)b}\right) \ \right|^q dt \\ &\leq \frac{1}{3}\left| \ f'(a) \ \right|^q \int_0^1 \left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)^{2q} \frac{1+t}{2} dt \\ &+ \frac{1}{3}\left| \ f'(b) \ \right|^q \int_0^1 \left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)^{2q} \frac{1-t}{2} dt\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 \left|\frac{1}{3}-\frac{t^\alpha}{2}\right| \left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)^{2q} \left| \ f'\left(\tfrac{2ab}{(1+t)a+(1-t)b}\right) \ \right|^q dt \\ &\leq \frac{1}{3}\left| \ f'(a) \ \right|^q \int_0^1 \left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)^{2q} \frac{1-t}{2} dt \\ &+ \frac{1}{3}\left| \ f'(b) \ \right|^q \int_0^1 \left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)^{2q} \frac{1+t}{2} dt.\end{aligned}$$

By simple computation,

$$\left(\int_0^1 \left|\frac{1}{3}-\frac{t^\alpha}{2}\right| dt\right)^{1-\frac{1}{q}} = \left[\frac{1}{3\,(\alpha+1)}\left(\left(\frac{2}{3}\right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha\right)\right]^{1-\frac{1}{q}}.$$

Using the last two inequalities and the equation, we obtain

$$\begin{aligned}&|I(a,b;\alpha)| \leq \frac{b-a}{6ab}\left[\frac{1}{\alpha+1}\left(\left(\frac{2}{3}\right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha\right)\right]^{1-\frac{1}{q}} \\ &\times \left[\begin{array}{c} \left(\left| \ f'(a) \ \right|^q \int_0^1 \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^{2q} \frac{1+t}{2} dt + \left| \ f'(b) \ \right|^q \int_0^1 \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^{2q} \frac{1-t}{2} dt\right)^{\frac{1}{q}} \\ + \left(\left| \ f'(a) \ \right|^q \int_0^1 \left(\frac{2ab}{(1+t)a+(1-t)b}\right)^{2q} \frac{1-t}{2} dt + \left| \ f'(b) \ \right|^q \int_0^1 \left(\frac{2ab}{(1+t)a+(1-t)b}\right)^{2q} \frac{1+t}{2} dt\right)^{\frac{1}{q}} \end{array}\right].\end{aligned}$$

This ends the proof. Q.E.D.

Remark 3.7. If we take $\alpha = 1$ in Theorem 3.6, we obtain the following inequality

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{2ab}{a+b}\right) + f(b) \right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \tag{3.6}$$
$$\leq \frac{b-a}{6ab}\left(\frac{5}{12}\right)^{1-\frac{1}{q}} \left[\begin{array}{c} \left(X_1(q,a,b) \left| f'(a) \right|^q + X_2(q,a,b) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \\ + \left(X_3(q,a,b) \left| f'(a) \right|^q + X_4(q,a,b) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \end{array} \right].$$

4 Conclusion

In this paper, using a new identity of Simpson-like type for Riemann-Liouville fractional integral for harmonic convex functions, we obtain some new Simpson type Riemann-Liouville fractional integral inequalities. Furthermore, some interesting conclusions is obtained for some special values of α. So, this paper is a detailed examination of the Simpson-like type Riemann-Liouville fractional integral inequalities.

References

[1] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam (2006).

[2] A.M. Latif, S. Hussein and M. Baloch, *Weighted Simpson's Type Inequalities for HA-convex*, Punjab University J. Math. **52** (7) (2020) 11–24.

[3] C. Peng, C. Zhou and T.S. Du, *Riemann-Liouville fractional Simpson's inequalities through generalized* $(m, h_1, h_2)-$*preinvexity*, Italian Journal of Pure and Applied Mathematics **38** (2020) 343–367.

[4] C.Y. Luo, T.S. Du and Y. Zhang, *Certain new bounds considering the weighted Simpson-like type inequality and applications*, J. Inequal. Appl. **2018** (2018) Article ID 332.

[5] G.N. Watson, *A Treatise Theory of Bessel Functions, Cambridge University Press*, Cambridge (1944).

[6] I. Iscan, *Hermite-Hadamard typeinequalities for harmonically convex functions*, Hacettepe J. Math. Statist. **46** (6) (2014) 935–942.

[7] M. Alomari, M. Darus and S.S. Dragomir, *New inequalities of Simpson's type for s-convex functions with applications*, RGMIA Res. Rep. Coll. **12** (4) (2009).

[8] M. Matloka, *Some inequalities of Simpson type for h-convex functions via fractional integrals*, Abstr. Appl. Anal. **2015** (2015) 5 pages.

[9] M. Matloka, *Weighted Simpson type inequalities for h-convex functions*, J. Nonlinear Sci. Appl. **10** (2017) 5570–5780.

[10] M. Tunç, E. Gov and S. Balgeçti, *Simpson type quantum integral inequalities for convex functions*, Miskolc Math. Notes **19** (1) (2018) 649–664.

[11] M.Z. Sarikaya and E. Set and E. Ozdemir, *On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex*, Journal of Applied Mathematics, Statistics and Informatics **9** (1) (2013) 37–45.

[12] M.Z. Sarikaya and E. Set and E. Ozdemir, *On new inequalities of Simpson's type for s-convex functions*, Comput. Math. Appl. **60** (2010) 2191–2199.

[13] M.Z. Sarikaya and S. Bardak, *Generalized Simpson type integral inequalties*, Konuralp J. Math. **7** (2019) 186–191.

[14] S. Rashid, A.O. Akdemir, F. Jarad, M.A. Noor and K.I. Noor, *Simpson's type integral inequalities for* $k-$*fractional integrals and their applications*, AIMS. Math. **4** (4) (2019) 1087–1100.

[15] S.S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998) 91–95.

[16] S.S. Dragomir, R.P. Agarwal and P. Cerone, *On Simpson's inequality and applications*, J. of Ineq. and Appl. **5** (2) (2000).

[17] T. Zhu, P. Wang and T.S. Du, *Some estimates on the weighted Simpson like type integral inequalities and their applications*, J. Nonlinear Funct. Anal. **2020** (2020) Article ID 17.

[18] Z. Sanli, M. Kunt and T. Koroglu, *New Riemann-Liouville fractional Hermite-Hadamard type inequalities for harmonically conwex functions*, Arab. J. Math. **9** (2020) 431–441.